**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**

# MATH2020B Advanced Calculus II
# Suggested Solutions for Homework 8
**Date: 3 April, 2025**

1. (a) Apply Green's theorem to evaluate

$$\oint_C y^2 dx + x^2 dy$$

where $C$ is the triangle bounded by $x = 0, x + y = 1, y = 0$.

(b) Do this for

$$\oint_C 3y dx + 2x dy$$

where $C$ is the boundary of $0 \leq x \leq \pi, 0 \leq y \leq \sin(x)$.

**Solution.** (a) Let $M = y^2, N = x^2$. Then since $C$ is a piecewise-smooth, simple closed curve, by Green's theorem we have that

$$\begin{aligned}
\oint_C y^2 dx + x^2 dy &= \int_0^1 \int_0^{1-x} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx \\
&= \int_0^1 \int_0^{1-x} (2x - 2y) dy dx \\
&= \int_0^1 (2xy - y^2) \Big|_{y=0}^{y=1-x} dx \\
&= \int_0^1 (-3x^2 + 4x - 1) dx \\
&= \left( -x^3 + 2x^2 - x \right) \Big|_{x=0}^{x=1} \\
&= 0.
\end{aligned}$$

(b) Similarly, let $M = 3y, N = 2x$, then Green's theorem gives

$$\begin{aligned}
\oint_C 3y dx + 2x dy &= \int_0^\pi \int_0^{\sin(x)} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx \\
&= \int_0^\pi \int_0^{\sin(x)} (2 - 3) dy dx \\
&= -\int_0^\pi \int_0^{\sin(x)} dy dx \\
&= -\int_0^\pi \sin(x) dx \\
&= \cos(x) \Big|_{x=0}^{x=\pi} \\
&= -2.
\end{aligned}$$

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2. Let $C$ be a simple closed curve on $\mathbb{R}^2$ with counterclockwise orientation. Suppose that the region $\Omega$ enclosed by $C$ is simply connected. Show that

$$\text{Area}(\Omega) = \frac{1}{2}\oint_C xdy - ydx.$$

Then use this to find the area of the ellipse

$$\{(x,y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}.$$

**Solution.** By Green's theorem, we have

$$\frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\iint_\Omega (1-(-1))dxdy = \frac{1}{2}\cdot 2\iint_\Omega dxdy = \text{Area}(\Omega).$$

Then applying this to the ellipse $C$, we parameterize $C$ by

$$\vec{r}(t) = (a\cos(t), b\sin(t)), \quad 0 \le t \le 2\pi.$$

Then $x = a\cos(t), dy = b\cos(t)dt, -y = -b\sin(t), dx = -a\sin(t)dt$ and using the formula above we have

$$\begin{aligned}
\text{Area(ellipse)} &= \frac{1}{2}\oint_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos(t)b\cos(t) + b\sin(t)a\sin(t))dt \\
&= \frac{1}{2}\int_0^{2\pi} ab(\cos^2(t) + \sin^2(t))dt \\
&= \pi ab.
\end{aligned}$$

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3. Find the area of the astroid enclosed by $\vec{r}(t) = \cos^3(t)\vec{i} + \sin^3(t)\vec{j}$ for $0 \le t \le 2\pi$.

**Solution.** Let $C$ be the curve parameterized by $\vec{r}(t)$. We use the formula established in the previous problem. We have $x = \cos^3(t), dy = 3\sin^2(t)\cos(t)dt, y = \sin^3(t), dx = -3\cos^2(t)\sin(t)dt$ and

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\oint_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} (3\cos^4(t)\sin^2(t) + 3\sin^4(t)\cos^2(t))dt \\
&= \frac{3}{2}\int_0^{2\pi} \cos^2(t)\sin^2(t)dt \\
&= \frac{3}{2}\int_0^{2\pi} \left(\frac{1+\cos(2t)}{2}\right)\left(\frac{1-\cos(2t)}{2}\right)dt \\
&= \frac{3}{2}\int_0^{2\pi} \left(\frac{1-\cos^2(2t)}{4}\right)dt \\
&= \frac{3}{2}\int_0^{2\pi} \left(\frac{1}{8} - \frac{\cos(4t)}{8}\right)dt \\
&= \frac{3\pi}{8}.
\end{aligned}$$

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4. Compute the area of the surface cut from the "nose" of the paraboloid $x = 1-y^2-z^2$ by the $yz$-plane.

**Solution.** Writing $f(y,z) = 1-y^2-z^2$, our surface is given by the graph $x = f(y,z)$ over the region $\Omega$ that is the region in the $yz$-plane when $x = 0 \Leftrightarrow 1 = y^2+z^2$, i.e. the disk of radius 1 in the $yz$-plane. Hence, using the formula for the surface area in Lecture notes 16, we have

$$\begin{aligned}
\text{Area} &= \iint_{y^2+z^2\le 1} \sqrt{f_y^2+f_z^2+1}\,dydz \\
&= \iint_{y^2+z^2\le 1} \sqrt{(-2y)^2+(-2z)^2+1}\,dydz \\
&= \iint_{y^2+z^2\le 1} \sqrt{4(y^2+z^2)+1}\,dydz \\
&= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\,rdrd\theta
\end{aligned}$$

switching to polar coordinates $r^2 = y^2+z^2$ in the last line above. By the change of variables $u = 4r^2+1, du = 8rdr$, we have

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\,rdrd\theta \\
&= \int_0^{2\pi}\int_1^5 \frac{1}{8}\sqrt{u}\,du \\
&= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=5} \\
&= \frac{\pi}{6}(5\sqrt{5}-1).
\end{aligned}$$

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5. Compute the surface area of the circular torus obtained by rotating the circle $C$ of radius $r$ and center $(R,0,0)$ on the $xz$-plane around the $z$-axis. Find a parameterization of the torus and compute its surface area.

**Solution.** The circle $C$ is given by $(x-R)^2+z^2 = r^2$ (where we need $0 < r < R$) and is parameterized by

$$(r\cos(u)+R, 0, r\sin(u)) \quad 0 \le u \le 2\pi.$$

Then rotating around the $z$-axis, we obtain the parameterization of the torus

$$\vec{r}(u,v) = ((r\cos(u)+R)\cos(v), (r\cos(u)+R)\sin(v), r\sin(u))$$

with

$$\begin{aligned}
\vec{r}_u &= (-r\sin(u)\cos(v), -r\sin(u)\sin(v), r\cos(u)) \\
\vec{r}_v &= (-(r\cos(u)+R)\sin(v), (r\cos(u)+R)\cos(v), 0)
\end{aligned}$$

and cross product

$$\begin{aligned}
\vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin(u)\cos(v) & -r\sin(u)\sin(v) & r\cos(u) \\ -(r\cos(u)+R)\sin(v) & (r\cos(u)+R)\cos(v) & 0 \end{vmatrix} \\
&= \left( \begin{vmatrix} -r\sin(u)\sin(v) & r\cos(u) \\ (r\cos(u)+R)\cos(v) & 0 \end{vmatrix}, - \begin{vmatrix} -r\sin(u)\cos(v) & r\cos(u) \\ -(r\cos(u)+R)\sin(v) & 0 \end{vmatrix}, \right. \\
&\quad \left. \begin{vmatrix} -r\sin(u)\cos(v) & -r\sin(u)\sin(v) \\ -(r\cos(u)+R)\sin(v) & (r\cos(u)+R)\cos(v) \end{vmatrix} \right) \\
&= (-r\cos(u)(r\cos(u)+R)\cos(v), \\
&\quad - r\cos(u)(r\cos(u)+R)\sin(v), -r\sin(u)(r\cos(u)+R))
\end{aligned}$$

which has length

$$\begin{aligned}
\|\vec{r}_u \times \vec{r}_v\| &= (r\cos(u)+R)\sqrt{r^2\cos^2(u)\cos^2(v) + r^2\cos^2(u)\sin^2(v) + r^2\sin^2(u)} \\
&= r(r\cos(u)+R).
\end{aligned}$$

Hence, the surface area is given by

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi}\int_0^{2\pi} r(r\cos(u)+R)dudv \\
&= \int_0^{2\pi} (r^2\sin(u) + rRu)\Big|_{u=0}^{u=2\pi} dv \\
&= \int_0^{2\pi} 2\pi rR dv \\
&= 4\pi^2 rR.
\end{aligned}$$

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